Journal of Applied Science, Engineering, Technology and Management
Vol. 01, Issue 02, December 2023 pp. 23 - 28

Inner product spaces and Fourier Series Approximation

Jill K. Mathew
Assistant Professor,
Department of Mathematics,
Mar Ivanios College (Autonomous),
Thiruvananthapuram, Kerala, India -695015.
jill.mathew@mic.ac.in

***Abstract* — The aim of this paper is to give the basic notions of the theory of Fourier Series in connection with Inner production spaces. The preliminary concepts and fundamentals of normed spaces, inner product spaces and the Fourier series are being discussed. The space of all periodic complex integrable functions forms an inner product space. Basic trigonometric functions $Sin\ x$ and $Cos\ x$ will form a basis for this linear space. A special emphasis is given in deriving the formulae for the Fourier coefficients. The convergence properties of Fourier series are also studied.**

Keywords — ***Inner product, Fourier series, Periodic functions, Convergence of series.***

I. Introduction

In this section, some of the fundamentals of linear spaces, normed spaces, inner product spaces, orthogonality in inner product spaces and Fourier series are given. All these are very essential for the next section on the class of periodic complex integrable functions and the Fourier development.

1.1 Linear spaces:

In this subsection, the basics and preliminaries of linear spaces, which are also called vector spaces are discussed. The main issue related with an infinite dimensional linear space is also presented here.

A linear space [1] over a field K is a non empty set V along with a function $+: V \times V \longrightarrow V$, called addition, and a function $\cdot : K \times V \longrightarrow V$, called scalar multiplication, such that

1. $(V, +)$ is an abelian group
2. $k \cdot (x + y) = k \cdot x + k \cdot y$, for all $k \in K$ and for all $x, y \in V$
3. $(k + l) \cdot x = k \cdot x + l \cdot x$, for all $k, l \in K$ and for all $x \in V$
4. $(kl) \cdot x = k \cdot (l \cdot x)$, for all $k, l \in K$ and for all $x \in V$
5. $1 \cdot x = x$

It is a best practice to write kx in place of $k \cdot x$. If $K = \mathbb{R}$, then V is called a real linear space and if $k = \mathbb{C}$, V is called a complex linear space. Elements of the linear space V are called vecors, and elements of the field K are called scalars.

A nonempty set E of the linear space V is said to be a subspace [1] of V, if $kx + ly \in E$ for all $x, y \in E$ and for all $k, l \in K$. For any nonzero finite subset E of V, span of E is denoted and defined by $span\ E = \{k_1x_1 + k_2x_2 + \cdots + k_nx_n: x_1, x_2, \dots, x_n \in E;\ k_1, k_2, \dots, k_n \in K\}$. One can easily prove that $span\ E$ is the smallest subspace of V containing E. If $span\ E = V$, we say that E spans V.

A nonzero finite subset E of V is said to be linearly independent [2] if for all $x_1, x_2, \dots, x_n \in E$ and for all $k_1, k_2, \dots, k_n \in K$, the equation $k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0$ implies that $k_1 = k_2 = \cdots = k_n = 0$. If E is not linearly independent, it is called linearly dependent. A linearly independent set cannot contain the zero vector.

A nonzero finite subset E of V is called a Hamel basis or simply basis of V, if $span\ E = V$ and E is linearly independent. There are two points which must be remembered in the context of basis of a linear space. One is on existence of a basis and the other is on the uniqueness of basis.

For the former one, the problem has an affirmative solution. That is every linear space possess a basis. The proof of this statement uses one of the most celebrated results of set theory, namely, the Zorn's lemma. The lemma states that, any nonempty partially ordered set in which every totally ordered set has an upper bound possess a maximal element.

The existence of such a maximal linearly independent subset of V can be guaranteed by Zorn's lemma. The simple procedure which could be applied is: start with a linearly independent subset of V, and progressively enlarge it until it spans V. One thing we notice is that, every nonempty maximal linearly independent subset will be a basis of V.

For the latter one, the solution is not affirmative. That is a basis can have more than one basis. Still there exists some

eISSN: 2584-0371

Received for review: 27-08-2023
Accepted for publication: 07-12-2023
Published: 15-12-2023

unique feature, even there are more than one. Suppose a linear space V has a basis consisting of n elements, $1 \leq n < \infty$. Then any other basis for V also possess the same number of elements.

The cardinality [3] of the basis of a linear space is called the dimension of the linear space. If the linear space V has a basis with finite number of elements, then V is called finite dimensional. Trivially the space $\{0\}$ has zero dimension. There are linear spaces which are infinite dimensional. A linear space V is called infinite dimensional if it contains a linearly independent subset. The spanning criterion is very much associated with the notion of convergence of an infinite series. After introducing the notion of norm, $\|\cdot\|$ on the linear space V, the issue of convergence will be rectified.

The next subsection on normed spaces helps us to understand the analytic nature of a linear space by introducing a metric through norm.

1.2 Normed spaces:

In a linear space, addition of two vectors and scaling of a given vector by a scalar are the two operations which are permitted to do. One could not able to measure the length of a given vector. By introducing a norm on a linear space, the length of each vector can easily be measured. In this subsection, the fundamentals of a normed space are presented. This notion will help to deal with the convergence of infinite series in an infinite dimensional linear space.

Let V be a linear space over a field K. A norm [3, 4], denoted by $\|\cdot\|$ is a function from V to $\mathbb{R}$ such that for all $x, y \in V$ and $k \in K$, the following conditions are satisfied.

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|x + y\| \leq \|x\| + \|y\|$
4. $\|\mathrm{kx}\| = |k|\|x\|$.

A linear space V with a norm equipped in it is called a normed space.

Introduction of a norm on a linear space is not enough to fulfil all the advantages which could be enjoyed in a metric space. From a norm, we extend the idea to measure the distance [4] between two given vectors. The distance between two vectors x and y is denoted by $d(x, y)$ and is defined as the length of a new vector $x - y$.

That is $d(x, y) = \|x - y\|$. One can easily verify that, this d satisfy all the axioms of a metric.

Thus, all normed spaces are metric spaces with respect to the induced metric $d(x, y) = \|x - y\|$, the right side can simply be evaluated in a normed space.

One question which could be forwarded in this direction is finding out the number of norms that an individual can define in a linear space. The answer to this question is many norms can be defined on any linear space. But the norms could be compared in connection with the associated metrics. The equivalency [5] of two norms $\|\cdot\|$ and $\|\cdot\|'$ on the linear space V is defined as follows. The norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|'$ if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|x\|' \leq \alpha\|x\|$ for all $x \in V$. One of the most beautiful results regarding equivalency of norms is that, in any finite dimensional linear space, all the norms are equivalent.

A complete normed space is a Banach space. Being complete in the sense that, all Cauchy sequences are convergent sequences. The concepts of Cauchy and convergent sequences, can be completely handled in a normed space through the induced metric. Studies in this direction are relevant, as there are normed spaces which are not Banach spaces. All finite dimensional normed spaces Banach [6], the discretion could be more suitable in infinite dimensional spaces.

The next subsection on inner product space helps us to understand the geometry of a linear space.

1.3 Inner product spaces:

In this subsection, the basic notions of inner product spaces are presented. These are very useful in dealing with geometric concepts like perpendicularity in the setting of a linear space. Moreover, this will be applied to Fourier analysis in the next chapter.

Let V be a complex linear space. A complex inner product on V is an operation $\langle \cdot, \cdot \rangle: V \times V \to K$, which satisfies the following for all $x, y, z \in V$ and $a, b \in \mathbb{C}$:

1. conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. linearity in the first term: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
3. non negativity: $\langle x, x \rangle \geq 0$
4. non degeneracy: $\langle x, x \rangle = 0$ if and only if $x = 0$.

Real inner products can be defined in the same manner. In this entire article, “inner product” means “complex inner product”. If the linear space V is equipped with an inner product, it is called an inner product space.

An example of an inner product is given below.

Let $V = \mathbb{C}^n = \{x = (x_1, x_2, \dots, x_n): x_i \in \mathbb{C},\ 1 \leq i \leq n\}$.

For $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n) \in V$,

define $\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y}_j$. If $V = \mathbb{R}^n$,

just take the coordinate wise product and then the summation, the conjugation has no meaning.

The following three fundamental results, namely, polarization identity, Schwarz inequality and the Parallelogram law play an important role in the theory of inner product spaces.

The polarization identity states that, for all $x, y \in V$, we have $\langle x,y\rangle = \frac{1}{4}[\langle x+y,x+y\rangle - \langle x-y,x-y\rangle + i\langle x+iy,x+iy\rangle - i\langle x-iy,x-iy\rangle]$.

This identity shows that an inner product on a linear space V, is completely determined by the diagonal entries $\langle z,z\rangle, z \in V$. Moreover, an element x is completely determined by the scalars $\langle x,y\rangle, y \in V$.

The Schwarz inequality states that, for all $x, y \in V$, $|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$. It can easily be proved that; the inequality becomes an equality when $\{x,y\}$ is linearly dependent.

Finally, the third result, namely the parallelogram law gives an interplay between the structural properties of the norm and the inner product of the linear space. The parallelogram law states that, for all $x, y \in X$, we have $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

An inner product $\langle\cdot,\ \cdot\rangle$ on a linear space V induces a norm $\|\cdot\|$ on V in the canonical way.

That is for $x \in V$, $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$.

Thus, all inner product spaces can be considered as normed spaces. Conversely, one can think about the implication from a normed space to an inner product space. If $\|\cdot\|$ is a norm on a linear space V, which satisfies the parallelogram law, then in accordance with polarization identity, we can easily define, $\langle x,y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2]$.

This result is due to Jordan and von Neumann, which actually characterizes all inner product spaces among all normed spaces.

An inner product space which is complete in the norm induced by the inner product is called a Hilbert space. All inner product spaces being normed spaces, it could easily be deduced that, all Hilbert spaces are Banach spaces. But for the converse, the result of Jordan and Neumann shows that, all Banach spaces that satisfy the parallelogram law are Hilbert spaces.

1.4 Orthogonality in inner product spaces:

Two vectors x and y in an inner product space V are called orthogonal if $\langle x,y\rangle = 0$. We write this as $x \perp y$. Two subsets E and F of the inner product space V are called orthogonal if $x \perp y$ for all $x \in E$ and for all $y \in F$. A subset E is said to be orthogonal if $x \perp y$ for all $x \neq y$ in E. An orthogonal set E is called orthonormal if $\|x\| = 1$ for all $x \in E$. Some trivial implications are mentioned below.

The zero vector can be a member of an orthogonal set. So, an orthogonal set can be a linearly dependent one. If an orthogonal set is free from the zero vector, it is linearly independent. In an orthonormal set, each member is of unit length, the zero vector cannot be included in an orthonormal set. An orthonormal set is always linearly independent. Conversely, a linearly independent set can be made orthogonal and in turn orthonormal by Gram- Schmidt Orthonormalization process.

The Pythagoras theorem holds good in an orthogonal set. If $\{x_1, x_2, x_3, \dots, x_n\}$ is an orthogonal set, then $\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2$. It is very interesting to note that, the distance between any two vectors in an orthonormal set is always $\sqrt{2}$. For a brief justification, pick any two members x and y from an arbitrary orthonormal set E. Now $\|x-y\|^2 = \langle x-y,x-y\rangle = \langle x,x\rangle + \langle x,-y\rangle - \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2 = 1+1 = 2$.

The notion of projections is an important geometrical feature of an orthonormal set. One can project a vector in a higher dimensional linear space onto a lower dimensional subspace. For example, an individual can project a vector in $\mathbb{R}^3$ onto $\mathbb{R}^2$. It is very clear from common logic that; the converse will not hold good. Formally, if W is a subspace of the linear space V, a projection of x onto W is a vector $y \in W$, where $x - y$ is orthogonal to every vector in W.

The following two propositions are of great importance in the development of Fourier series. The first one is regarding the projection of a vector onto the span of an orthonormal set and the second one says that projection to the span of an orthonormal set is always a better approximate than any other vector in the span of the orthonormal set.

Proposition 1: Let $E = \{e_1, e_2, \dots, e_n\}$ be an orthonormal subset of an inner product space V. Let $x \in V$ be arbitrary. Define $c_n = \langle x, e_n\rangle$ and $s = \sum_{j=1}^{n} c_n e_n$. Then s is a projection of x onto the span of E.

The proof is very simple by applying the definition of projection. We complete the proof by proving $\langle x-s, y\rangle = 0$, where y is an arbitrary vector in the span of E.

Proposition 2: Let $E = \{e_1, e_2, \dots, e_n\}$ be an orthonormal subset of an inner product space V. Let $x \in V$ be arbitrary. Define $c_n = \langle x, e_n\rangle$ and $s = \sum_{j=1}^{n} c_n e_n$. Then for any y in the span of E, we have $\|x-s\| \le \|x-y\|$.

The next subsection gives us a brief outline of Fourier series.

1.5 Fourier Series:

In this subsection, only an overview of Fourier series is presented, just to know how the expression looks like. In the next chapter, the class of complex periodic integrable functions could be dealt with.

A Fourier series [7] is an expression of a periodic function in terms of an infinite sum of $sines$ and $cosines$. It strongly depends upon the orthogonal relationship of $sine$ and $cosine$ functions. The computation and study of Fourier series is known as harmonic analysis.

Suppose that $f(x)$ is a periodic function with period 2π. Then the real form of the Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k\, sinkx)\,; k = 0, 1, 2, \ldots,$ where $a_k = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\ Cos\ kx\, dx$ and

$$b_k = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\ Sin\ kx\ dx.$$

The complex form of the Fourier series of $f(x)$ is given by $f(x) = \sum_{-\infty}^{\infty} C_k e^{ikx}$, where $c_k = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, e^{-ikx}\, dx$.

In the next chapter, it could easily be seen that, how the above formulae could be derived by using the help of an inner product.

II. Inner product in Fourier series

In this chapter, the properties of the space of all complex periodic integrable functions are discussed in detail. This space is equipped with an inner product, and later, it could be found out that, the Fourier coefficients are actually the inner products of the function and *sines* or *cosines*. Some relations between the norms of functions and their Fourier coefficients, provided by Bessel's inequality and Parseval's identity are also being investigated.

2.1 The inner product space of complex periodic integrable functions:

Here, it is proved that, the set of all complex, periodic, Riemann integrable functions forms a linear space and then an inner product space.

Theorem 2.1.1: Let V^* be the set of all 2π −periodic complex valued Riemann integrable functions. Clearly V^* is a linear space under usual addition of functions and scaling of functions by complex numbers. The inner product on V^* is defined as follows.

For $f, g \in V$, define $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\ dx$.

Let us verify that this definition satisfies all the axioms of an inner product.

To prove the conjugate symmetry,

set $f(x) = a(x) + ib(x)$ and $g(x) = c(x) + id(x)$.

Now,$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\ dx = \frac{1}{2\pi}\int_0^{2\pi}[a(x) + ib(x)][c(x) - id(x)]\ dx,$

after some manipulations we see that $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi}[c(x) + id(x)][a(x) - ib(x)]\ dx = \frac{1}{2\pi}\int_0^{2\pi} g(x)\overline{f(x)}\ dx = \overline{\langle g, f\rangle}$.

It is easy to prove the linearity in the first variable, for, $\langle \alpha f + \beta g, h\rangle = \frac{1}{2\pi}\int_0^{2\pi}[\alpha f(x) + \beta g(x)]\overline{h(x)}\ dx = \frac{\alpha}{2\pi}\int_0^{2\pi} f(x)\overline{h(x)}\ dx + \frac{\beta}{2\pi}\int_0^{2\pi} g(x)\overline{h(x)}\ dx = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$.

It can also be seen that the positive definiteness holds good in this definition.

For, $\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{f(x)}\ dx = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\ dx \geq 0$.

Finally, we verify the non-degeneracy, if $f = 0$, then $\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} 0\ dx = 0$.

Conversely, $\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{f(x)}\ dx = 0$ implies that $\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\ dx = 0$, which implies that $f = 0$. Thus $f = 0$ if and only if $\langle f, f\rangle = 0$.

It is remarked that, the norm induced by this inner product is given by

$\|f\|^2 = \langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{f(x)}\ dx = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\ dx$. With respect to this norm, we will deal with the convergence of the Fourier series.

In the next subsection, an orthonormal family will be identified in the inner product space which is already constructed.

2.2 Orthonormal set becomes a basis of the inner product space:

Here the basic requirements for the complex form of the Fourier series are identified.

Theorem 2.2.1: The family $\{(e_n)_{n\in\mathbb{Z}}\}$, defined by $e_n(x) = e^{inx} = Cos\ nx + i\ Sin\ nx$ is an orthonormal subset of the inner product space, V^*. The verification of this result as follows.

For $m, n \in \mathbb{Z}$, $\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{imx}\overline{e^{inx}}\ dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)x}\ dx = \begin{cases}1; m = n\\ 0; m \neq n\end{cases}$.

The key idea of the Fourier series is that, the family $\{(e_n)_{n\in\mathbb{Z}}\}$ is a basis for V^*. This means, every complex, periodic, Riemann integrable function can be uniquely represented as a linear combination of *sines* and *cosines*.

In the next result, it could be cleared that the sum of inner products of an arbitrary function f from V^* with any orthonormal set $\{(a_n)_{n\in\mathbb{Z}}\}$ is the projection of f onto the span of $\{(a_n)_{n\in\mathbb{Z}}\}$. By the best approximating property of projections, it is guarantee that, the convergence of the Fourier series is faster than any other.

Theorem 2.2.2: Let $f \in V^*$. Define $c_n = \langle f, a_n\rangle$, Let $t_N(f) = \sum_{|n|\leq N} c_n a_n$. Then $t_N(f)$ is the projection of f onto the span of $\{(a_n)_{n\in\mathbb{Z}}\}$.

The proof of this theorem is by using the definition of projections.

Let $a_m \in \{a_n\}_{|n|\leq N}$ be arbitrary. It is easy to prove that $\langle f - t_N(f), a_m\rangle = \langle f - \sum_{|n|\leq N} c_n a_n, a_m\rangle = \langle f, a_m\rangle - \sum_{|n|\leq N} c_n \langle a_n, a_m\rangle = \langle f, a_m\rangle - a_m = a_m - a_m = 0$.

The next result (the Bessel's inequality) plays an important role in the convergence of Fourier series [8]. It relates the magnitude of the Fourier coefficients and the norm of the function which one approximate.

Theorem 2.2.3: Let $f \in V^*$ be an arbitrary function and suppose that $\{a_n : n \in \mathbb{Z}\}$ be an arbitrary orthonormal subset of V^*. Let $c_n = \langle f,\ a_n \rangle$. Then $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \|f\|^2$.

Proof: As in the proof of theorem 2.2.2, let $t_N(f) = \sum_{|n|\leq N} c_n a_n$. Since, $t_N(f)$ is a member of the span of $\{a_n\}_{|n|\leq N}$, we have, $t_N(f)$ is orthogonal to $f - t_N(f)$. So, we get $\|f\|^2 = \|f - t_N(f) + t_N(f)\|^2 = \|f - t_N(f)\|^2 + \|t_N(f)\|^2$.

But by easy manipulations, we can get

$$\|t_N(f)\|^2 = \sum_{|n|\leq N} |c_n|^2.$$

So, the above equation becomes,

$$\|f\|^2 = \|f - t_N(f)\|^2 + \sum_{|n|\leq N} |c_n|^2.$$

From this we get, $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \|f\|^2$.

III. Convergence of the Fourier series

In this chapter, it is proved that the Fourier series is the best approximation to any function in V^* than any other combination of vectors in any orthonormal set in V^*. In the following subsection, this fact is revealed.

3.1 Best approximation:

In the following theorem, an inequality showing that the partial sum [8] of the Fourier series of f gives a better approximation to f than any linear combination of vectors which include non-Fourier coefficients, is presented.

Theorem 3.1.2: Let $f \in V^*$ be an arbitrary function with Fourier coefficients a_n. If $a_n = \langle f,\ e_n \rangle$, where $\{(e_n)_{n\in\mathbb{Z}}\}$, defined by $e_n(x) = e^{inx} = Cos\, nx + i\, Sin\, nx$. Then $\|f - s_N(f)\| \leq \left\|f - \sum_{|n|\leq N} c_n e_n\right\|$ for any $c_n \in \mathbb{C}$.

Proof: By simple addition and subtraction, get $\sum_{|n|\leq N} c_n e_n = \sum_{|n|\leq N}(a_n e_n + c_n e_n - a_n e_n)$

$= \sum_{|n|\leq N} a_n e_n - \sum_{|n|\leq N}(a_n e_n - c_n e_n) = s_N(f) - \sum_{|n|\leq N}(a_n - c_n)e_n$

This implies that

$$\left\|f - \sum_{|n|\leq N} c_n e_n\right\|^2 = \left\|f - s_N(f) + \sum_{|n|\leq N}(a_n - c_n)e_n\right\|^2$$

By orthogonality of $f - s_N(f)$ to $\sum_{|n|\leq N}(a_n - c_n)e_n$, get

$$\left\|f - s_N(f) + \sum_{|n|\leq N}(a_n - c_n)e_n\right\|^2 = \|f - s_N(f)\|^2 + \left\|\sum_{|n|\leq N}(a_n - c_n)e_n\right\|^2.$$

But this expression is greater than or equal to $\|f - s_N(f)\|^2$. Thus $\|f - s_N(f)\| \leq \left\|f - \sum_{|n|\leq N} c_n e_n\right\|$ as desired.

This theorem says that the partial sum of the Fourier series of f gives a better approximation to f than any linear combination of vectors in $\{c_n\}_{n\in\mathbb{Z}}$ of non-Fourier coefficients.

The following subsection deals with the most useful property of the Fourier series, namely the Parseval's identity. This identity establishes the relation between sum of the squares of the magnitudes of the Fourier coefficients and the square of the norm of the approximating function.

3.2 Parseval's Identity:

In the first result, it is showed that the Fourier series of f approximates f with increasing accuracy as the degree of the term approaches infinity.

Theorem 3.2.1: Suppose $f \in V^*$ be an arbitrary function with Fourier coefficients $a_n = \langle f, e_n \rangle$. Then $\lim_{N\to\infty} \|f - s_N(f)\| = 0$.

The entire proof of this result is not presented here. One could prove the result by considering two cases based on the continuity of f. We also use the following result to prove the case when f is continuous.

Result: Suppose f is a continuous 2π −periodic function. Then for all $\varepsilon > 0$, there exists a function of the form $P(x) = \sum_{-M}^{M} c_n e^{inx}$ with $c_n \in \mathbb{C}$ such that $|f(x) - P(x)| < \varepsilon$ for all x.

Now the background is ready to establish the Parseval's identity.

Theorem 3.2.2: If $f \in V^*$, then $\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2$.

The proof is clear from the previous theorems (Theorem 3.1.2 and Theorem 3.2.1)

IV. Conclusion

In this article, a journey from Linear Algebra to the Fourier series is presented. A description of the consequence and importance of Zorn's lemma in Linear Algebra along with the basics and need of normed spaces and inner product spaces, was discussed in detail. The issue related to the convergence of infinite linear combinations in infinite-dimensional linear spaces is addressed in detail. After discussing the feature of orthogonality in inner product spaces, it is found out that, integrable functions forms an inner product space. An orthonormal set in the new inner product space is constructed and then defined the Fourier coefficients. Finally, the best approximating property of the Fourier series to the approximating function is proved. The best approximating property along with the Parseval's identity is a solution for the problem of convergence of the Fourier series.

References

[1] Halmos P. R., *Finite Dimensional Vector Spaces*, Springer, 1974.

[2] Kapalansky I., *Linear ALgebra and Geometry: A Second Course*, Dover, 2003.

[3] Bishop E., *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[4] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, Inc.1976.

[5] Balmohan V Limaye, *Functional Analysis (Revised third edition)*, New Age International, 2017.

[6] Diestel J., *Sequences and Series in Banach Spaces*, Springer – Verlag, 1984

[7] Elias M. Stein and Rami Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[8] Carleson A., *On the convergence and growth of partial sums of Fourier series*, Acta Mathematica, 116, pp 135 – 157, 1966.